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# Parabolic Itô Equations with Monotone Nonlinearities

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In this paper the equation $u_t = Lu - F(u) + \alpha(t, \omega)$ is studied, where $u(t) \in B_0$ a Banach space. $L$ is an unbounded self-adjoint negative definite operator. $F$ is a monotone nonlinear potential operator. $\alpha(t, \omega)$ is a *white noise* process on $B_0$. With suitable further restrictions on $L$ and $F$ it is proved that the equation has a unique solution. As $t \to \infty$ the distribution of $u(t, \omega)$ approaches a stationary distribution which is calculated explicitly.

Marcus [1] studied a parabolic Itô equation of the following form.

$$\frac{\partial u}{\partial t}(t, \omega) = Lu(t, \omega) - F(u(t, \omega)) + \alpha(t, \omega), \tag{1}$$

where $u(t, \omega) \in H$ is a Hilbert space; $L$ is a negative definite operator on $H$ with an inverse of finite trace; $\alpha(t, \omega)$ is a *white noise* on $H$, i.e., a Gaussian process of mean 0 and satisfying the heuristic formula

$$E_\omega\{(u, \alpha(t, \omega))(v, \alpha(s, \omega))\} = (u, v)\, \delta(t - s)$$

for all $u, v \in H$, where $(\cdot, \cdot)$ is the scalar product and $E_\omega$ is the expectation; and $F$ is a Lipschitz continuous operator on $H$ such that $L^{-1}F$ is a contraction.

It was proved that (1) has a unique solution $u(t, \omega)$ satisfying $\sup_t E_\omega\{\| u(t, \omega)\|^2\} < \infty$ where $\| \cdot \|$ is the norm in $H$. Also $u(t, \omega) = R(t, \omega) + V(t, \omega)$ where $R(t, \omega)$ is a stationary process and $\lim_{t\to\infty} E_\omega\{\| V(t, \omega)\|^2\} = 0$.

In addition if $F$ is the Fréchet derivative of some functional $f$ defined on $H$ and $L$ is self-adjoint; the distribution of $R(t, \omega)$ as $t \to \infty$ was calculated. It was shown to have a Radon–Nikodym derivative proportional to $\exp(-2f)$ with respect to the Gaussian measure on $H$ with mean 0 and covariance $-L^{-1}/2$.

In this paper the above results will be extended to an equation where the nonlinear term satisfies a monotonicity condition. The motivation for this is twofold. From the point of view of nonlinear evolution equations the results are a partial generalization of earlier work [2]. The more important purpose of this article is to lay the groundwork for a study of Markov fields. See the Appendix for further details.



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The equation studied is formally:

$$\frac{\partial u}{\partial t}(t, \omega) = Lu(t, \omega) - F(u(t, \omega)) + \alpha(t, \omega). \tag{2}$$

$u(0, \omega) \in B_0$, where $B_0$ is a reflexive Banach space with norm $|\cdot|_0$ which is a dense subset of its dual space $B^*_0$. Also $|\cdot|_0^*$ the norm of $B^*_0$ is assumed to be Fréchet differentiable. Let $H_0$ be the Hilbert space completion of $B_0$ with scalar product $(\cdot, \cdot)$ equal to the dual pairing $[\cdot, \cdot]$ of $B^*_0$ and $B_0$ and corresponding norm $\|\cdot\|_0$.

Condition (H): $\|u\|_0 \leqslant h\, |u|_0$ for some $h \geqslant 0$ and all $u \in B_0$.

$\alpha(t, \omega)$ is a *white noise* process on $B_0$, i.e., a Gaussian process of mean 0 satisfying the heuristic equation

$$E\{[\alpha(t, \omega), u]_0\, [\alpha(s, \omega), v]_0\} = [u, v]_0\, \delta(t - s) \qquad \text{with} \quad u, v \in B_0.$$

$F$ satisfies the following conditions:

(F1) $F: B_0 \to B^*_0$.

(F2) $F$ is demicontinuous; i.e., if $u_N \to u$ strongly then $F(u_N) \to F(u)$ weakly.

(F3) $F$ is uniformly bounded, i.e., $|F(u)|_0^* \leqslant a\, |u|_0^{p-1}$ for some $a \geqslant 0$ and $p \geqslant 2$.

(F4) $F$ is uniformly monotone; i.e.,

$$[F(u) - F(v), u - v]_0 \geqslant c\, |u - v|_0^p$$

for some $c > 0$ and all $u, v \in B_0$.

$L$ satisfies the following conditions:

(L1) $L$ is a densely defined self-adjoint negative definite operator on $H$.

(L2) $L^{-1}: B^*_0 \to B_0$ and is bounded.

(L3) $L$ has a complete set of eigenvectors $\phi_i \in B_0$ orthonrmal in $H$ with a decreasing set of eigenvalues, $\lambda_i$, such that $\sum_{i=0}^{\infty} |\phi_i|^2/\lambda_i < \infty$.

Note that $L$ is the infinitesmal generator of a semigroup of operators $G(t)$ satisfying:

(G1) $G(t)$: $B^*_0 \to B_0$ and is bounded.

(G2) $G(t)$ is positive, i.e., $[u, G(t)u]_0 \geqslant 0$ for all $u \in B^*_0$.

(G3)
$$\int_0^\infty \operatorname{Trace}(G^*(t)\, G(t))\, dt = \int_0^\infty \sum_{i=0}^{\infty} (G(t)\,\phi_i, G(t)\,\phi_i)\, dt = \sum_{i=0}^{\infty} 1/2\lambda_i < \infty.$$

Let $W(t, \omega)$ be a Gaussian process with state space $B_0$, mean $G(t)\, u(0)$, and covariance $\int_0^{\min(t,T)} G^*(t-s)\, G(T-s)\, ds$. Hence from (G3)

$$E_\omega\{(W(t), W(T))\} = \int_0^{\min(t,T)} \mathrm{Trace}(G^*(t-s)\, G(T-s))\, ds + E_\omega\{(G(t)\, u(0), G(t)\, u(0))\} < \infty.$$

Note that formally

$$W(t, \omega) = \int_0^t G(t-s)\, \alpha(s, \omega)\, ds + G(t)\, u(0).$$

Using (L3) a straightforward estimate shows that $W \in L^2(0, T; B_0)$ almost surely. Since $W$ is a Gaussian process this implies that $W \in L^p(0, T; B_0)$ almost surely for any $p \geqslant 2$. Let $B$ denote $L^p(0, T; B_0)$ and $B^*$ its dual with norm $|\cdot|$ and $|\cdot|^*$ respectively and dual pairing $[\cdot, \cdot]$. Let $H$ denote the corresponding Hilbert space with norm $\|\cdot\|$.

One can also verify from (L2), (G1) and (G2) that $G(t-s)$ satisfies:

(G4) $\int_0^t G(t-s)\, u(s)\, ds \in B$ and is bounded if $u \in B^*$.

(G5) $\left[u(t), \int_0^t G(t-s)\, u(s)\, ds\right] \geqslant 0$ for all $u \in B^*$.

Rewrite (2) as an integral equation:

$$u(t, \omega) = -\int_0^t G(t-s)\, F(u(s, \omega))\, ds + W(t, \omega). \tag{3}$$

THEOREM 1. *If $F$, $G$, and $W$ are as described above then there exists a unique $u \in B$ satisfying* (3) *such that $|u| \leqslant b\, |W|$ for some constant $b > 0$ almost surely.*

*Proof.* Let $Q \equiv u - W$. Rewrite (3) as an equation for $Q$. $Q = -\int_0^t G(t-s)\, F(Q + W)\, ds$. Note that as an operator on $Q \in B$, $F(Q + W)$ is almost surely bounded and demicontinuous from (F2), (F3), and the fact that $W \in B$ almost surely.

In addition there almost surely exists a constant $k(\omega) > 0$, such that $[F(Q + W), Q] > 0$ if $|Q| > k$. This follows because the uniform monotonicity condition (F4) implies $[F(Q + W) - F(W), Q] \geqslant c\, |Q|^p$. Hence $[F(Q + W), Q] \geqslant c\, |Q|^p + [F(W), Q] \geqslant c\, |Q|^p - a\, |W|^{p-1}\, |Q|$ by (F3). But $c\, |Q|^p - a\, |W|^{p-1}\, |Q| > 0$ for $|Q| > r\, |W|$, where $r = (a/c)$.

Now Theorem 19.1 of Vainberg [3] is applicable and guarantees the existence of a unique solution $u$ of (3). Furthermore the same theorem asserts that $|Q| \leqslant r\, |W|$ and hence

$$|u| = |Q + W| \leqslant |Q| + |W| \leqslant (1 + r)\, |W| \equiv b\, |W|.$$

Thus Theorem 1 has been proved.

It is possible to obtain a sequence of Galerkin approximations converging to $u$. Let $J_{0N}$ and $J_{0N}^*$ be the projection of $B_0$ and $B^*{}_0$ onto the subspaces spanned by the first $N$ eigenvectors of $L$. Let $J_N$ and $J^*{}_N$ be the corresponding projections for $B$ and $B^*$, i.e., $(J_N u)(t) = J_{0N}(u(t))$. Then Theorem 23.4 of [3] proves the existence of a sequence of approximations $\hat{Q}_N \in B$ with $|\hat{Q}_N| \leqslant b\,|W|$ satisfying $\hat{Q}_N(t) = -J_N \int_0^t G(t-s)\, J^*{}_N F(J_N \hat{Q}_N + W))\,ds$ such that $\lim_{N\to\infty} |Q - \hat{Q}_N| = 0$.

Let $\hat{u}_N(t,\omega) = \hat{Q}_N(t,\omega) + W(t,\omega)$; then $\hat{u}_N(t,\omega)$ satisfies $\hat{u}_N(t,\omega) = -J_N \int_0^t G(t-s)\, J^*{}_N F(\hat{u}_N)\,ds + W(t)$.

In the case when $F$ is the Fréchet derivative of a functional $f$ it is possible to obtain explicit information about the distribution of $u(t)$ as $t \to \infty$. However, a slightly different approximating sequence must be used because $J^*{}_N F$ is not necessarily an exact Fréchet derivative.

Let $Q_N$ be the unique solution of $Q_N = -J_N \int_0^t G(t-s)\, J^*{}_N F(Q_N + J_N W)\,ds$ which exists and is bounded by an argument similar to the proof of Theorem 1. Let $u_N = Q_N + W$ and it follows that $J_N u_N = J_N Q_N + J_N W = Q_N + J_N W$ and hence:

$$u_N(t) = -J_N \int_0^t G(t-s)\, J_N{}^* F(J_N u_N)\,ds + W.$$

Note that $J^*{}_N F(J_N u_N)$ is uniformly monotone and is the Fréchet derivative of $f(J_N u_N)$. Before showing that $\hat{u}_N$ and $u_N$ converge to the same limit a preliminary result is needed.

LEMMA 1. *Weak* $\lim_{N\to\infty} (J^*{}_N F(\hat{u}_N) - J^*{}_N F(J_N \hat{u}_N)) = 0$.

*Proof.* Recall that $F$ is demicontinuous. Since

$$\lim_{N\to\infty}(J_N \hat{u}_N - \hat{u}_N) = \lim_{N\to\infty}(W - J_N W) = 0,$$

weak $\lim_{N\to\infty}(F(\hat{u}_N) - F(J_N \hat{u}_N)) = 0$. Since $J^*{}_N$ are projections, the lemma is proved.

THEOREM 2. $\lim_{N\to\infty} |u_N - \hat{u}_N| = 0$ *and hence* $\lim_{N\to\infty} u_N = u$.

*Proof.*

$$\begin{aligned} u_N - \hat{u}_N &= -J_N \int_0^t G(t-s)(J^*{}_N F(J_N u_N) - J^*{}_N F(\hat{u}_N))\,ds \\ &= -J_N \int_0^t G(t-s)(J^*{}_N F(J_N u_N) - J^*{}_N F(\hat{u}_N))\,ds \\ &\quad - J_N \int_0^t G(t-s)(J^*{}_N F(J_N \hat{u}_N) - J^*{}_N F(\hat{u}_N))\,ds. \end{aligned}$$

Hence

$$
\begin{aligned}
&[J^*_N F(J_N u_N) - J^*_N F(J_N \hat{u}_N), u_N - \hat{u}_N] \\
&\qquad + \Big[J^*_N F(J_N u_N) - J^*_N F(J_N \hat{u}_N), \\
&\qquad\qquad J_N \int_0^t G(t-s)(J^*_N F(J_N u_N) - J^*_N F(J_N \hat{u}_N))\, ds\Big] \\
&\quad = -\Big[J^*_N F(J_N u_N) - J^*_N F(J_N \hat{u}_N), \\
&\qquad\qquad J_N \int_0^t G(t-s)(J^*_N F(J_N \hat{u}_N) - J^*_N F(\hat{u}_N))\, ds\Big].
\end{aligned} \tag{4}
$$

But

$$
\begin{aligned}
&[J^*_N F(J_N u_N) - J^*_N F(J_N \hat{u}_N), u_N - \hat{u}_N] \\
&\quad = [F(J_N u_N) - F(J_N \hat{u}_N), J_N u_N - J_N \hat{u}_N] > c \mid J_N u_N - J_N \hat{u}_N \mid^p.
\end{aligned}
$$

The second term on the left-hand side of (4) is positive from (G2). Thus the right-hand side of (4) satisfies

$$
\begin{aligned}
&-\Big[J^*_N F(J_N u_N) - J^*_N F(J_N \hat{u}_N), \int_0^t G(t-s)(J^*_N F(J_N \hat{u}_N) - J^*_N F(\hat{u}_N))\, ds\Big] \\
&\quad \geqslant c \mid J_N u_N - J_N \hat{u}_N \mid^p.
\end{aligned} \tag{5}
$$

From Lemma 1 and uniform boundedness of $u_N$, $\hat{u}_N$, $G$ and $F$ it follows that the left-hand side of (5) goes to 0 as $N \to \infty$. Hence

$$\lim_{N\to\infty} \mid J_N u_N - J_N \hat{u}_N \mid = 0.$$

But

$$u_N - J_N u_N = \hat{u}_N - J_N \hat{u}_N = W - J_N W;$$

hence

$$\lim_{N\to\infty} \mid u_N - \hat{u}_N \mid = \lim_{N\to\infty} \mid J_N u_N - J_N \hat{u}_N \mid = 0,$$

and the proof of Theorem 2 is completed.

Now the behavior of $u(t, \omega)$ as $t \to \infty$ can be studied using the approximating sequence $u_N(t, \omega)$.

DEFINITION 1. A stochastic process $u(t, \omega)$ is asymptotically stationary (asym. stat.) in $B$ if $u(t, \omega) = R(t, \omega) + V(t, \omega)$ where $R(t, \omega)$ is a stationary process on $B_0$ and $V$ is transient, i.e., $\lim_{t\to\infty} \mid V(t)|_0 = 0$ almost surely.

LEMMA 2. *$W(t, \omega)$ is asym. stat. in $B$.*

*Proof.* $W(t, \omega)$ is a Gaussian process with mean $G(t)\, u(0)$ and covariance $\int_0^{\min(t,T)} G^*(t-s)\, G(T-s)\, ds$. Using (L2) and (L3) it is easy to verify the lemma.

Let $B_s$ be the Banach space of asym. stat. processes $u(\cdot, \omega) \in B$ with norm $| u |_s = (E_\omega\{| u |^p\})^{1/p}$.

LEMMA 3. *$Q \in B_s$ implies*

$$-J_N \int_0^t G(t-s)\, J^*{}_N F(J_N Q + J_N W)\, ds \in B_s .$$

*Proof.* $J_N W$ is asym. stat. from Lemma 2 hence $(J_N Q + J_N W) \in B_s$. Since $F$ does not depend explicitly on $t$ or $\omega$ and is a bounded demicontinuous functional, $J^*{}_N F(J_N Q + J_N W)$ is the sum of a bounded stationary process on $J^*{}_N B^*{}_0$ and one which goes weakly to 0 as $t \to \infty$. Since $J^*{}_N B^*{}_0$ is finite dimensional, weak convergence to 0 is equivalent to strong convergence. Condition (G1) can be used to complete the proof of the lemma.

LEMMA 4. *$u_N(t, \omega)$ is asym. stat.*

*Proof.* Let $u_N = Q_N + W$. $W$ is asym. stat. by Lemma 2. Recall that $Q_N(t, \omega)$ satisfies:

$$Q_N = -J_N \int_0^t G(t-s)\, J^*{}_N F(J_N Q_N + J_N W)\, ds. \tag{6}$$

From Lemma 3 this equation is meaningful for $Q_N \in B_s$. In addition, $F$ and $G$ still satisfy in $B_s$ the same continuity, boundedness, monotonicity, and positivity conditions as in Theorem 1. Hence Theorem 19.1 of [3] guarantees a unique bounded solution in $B_s$ for Eq. (6).

Since $u_N = Q_N + W$ with both $Q_N$ and $W$ asym. stat., $u_N$ is asym. stat. and by definition $u_N(t, \omega) = R_N(t, \omega) + V_N(t, \omega)$ where $R_N$ is stationary and $V_N$ is transient.

LEMMA 5. *For each $N$, $E_\omega\{\| V_N \|^2\} = E_\omega\{\int_0^T \| V_N \|_0^2\, ds\} < a_N$ where $a_N$ is independent of $T$.*

*Proof.* Let $W_N(t) = W(t) - G(t)\, W(0) + G(t)\, R_N(0)$. It follows that $R_N = -J_N \int_0^t G(t-s)\, J^*{}_N F(J_N R_N)\, ds + W_N$. Recall

$$u_N = -J_N \int_0^t G(t-s)\, J^*{}_N F(J_N u_N)\, ds + W.$$

Then

$$\begin{aligned} V_N &= u_N - R_N \\ &= -J_N \int_0^t G(t-s)(J^*{}_N F(J_N u_N) - J^*{}_N F(J_N R_N))\, ds + W - W_N . \end{aligned}$$

Hence

$$J_N V_N = -J_N \int_0^t G(t-s)(J^*{}_N F(J_N u_N) - J^*{}_N F(J_N R_N))\, ds + J_N W - J_N W_N .$$

This can be converted into a differential equation

$$\frac{d}{dt}(J_N V_N) = L J_N V_N - J^*{}_N F(J_N u_N) + J^*{}_N F(J_N R_N)$$

with the initial condition

$$J_N V_N(0) = J_N W(0) - J_N W_N(0) = J_N u(0) - J_N R_N(0).$$

The last equality follows from the definitions of $W_N$ and $W$.

The differential equation implies using (L1), (L3), and (F4).

$$\begin{aligned}
&\frac{d}{dt} \| J_N V_N \|_0^2 \\
&\quad = \frac{d}{dt}[J_N V_N, J_N V_N]_0 = 2\left[\frac{d}{dt}(J_N V_N), J_N V_N\right]_0 \\
&\quad = 2[L J_N V_N, J_N V_N]_0 - 2[J^*{}_N F(J_N u_N) - J^*{}_N F(J_N R), J_N u_N - J_N R_N]_0 \\
&\quad \leqslant -2\lambda_0 \| J_N V_N \|_0^2 - 2c \mid J_N V_N \mid_0^p \leqslant -2\lambda_0 \| J_N V_N \|_0^2 .
\end{aligned}$$

Hence

$$\begin{aligned}
\| J_N V_N(t)\|_0^2 &\leqslant \exp(-2\lambda_0 t) \| J_N u(0) - J_N R_N(0)\|_0^2 \\
&\leqslant \exp(-2\lambda_0 t) \| u(0) - R_N(0)\|_0^2
\end{aligned}$$

and

$$E_\omega\{\| J_N V_N(t)\|_0^2\} \leqslant \exp(-2\lambda_0 t)\, E_\omega\{\| u(0) - R_N(0)\|_0^2\}.$$

Thus

$$E_\omega\{\| J_N V_N \|^2\} = E_\omega \left\{\int_0^T \| J_N V_N(t)\|_0^2\, dt\right\} \leqslant E_\omega\{\| u(0) - R_N(0)\|_0^2\}/2\lambda_0 .$$

But by condition (H), Lemma 4, and Definition 1,

$$\begin{aligned}
E_\omega\{\| u(0) - R_N(0)\|_0^2\} &\leqslant 2E_\omega\{\| u(0)\|_0^2\} + 2E_\omega\{\| R_N(0)\|_0^2\} \\
&\leqslant 2E_\omega\{h^2 \mid u(0)|_0^2\} + 2E_\omega\{h^2 \mid R_N(0)|_0^2\} < \infty.
\end{aligned}$$

This means $E_\omega\{\| J_N V_N \|^2\}$ is bounded for each $N$ uniformly in $T$. To finish the proof of the lemma,

$$\begin{aligned}
V_N - J_N V_N &= W - W_N - J_N(W - W_N) \\
&= G(t)\,(W(0) - R_N(0) - J_N(W(0) - R_N(0))).
\end{aligned}$$

Hence

$$E_\omega\{\| V_N - J_N V_N \|_{0}^{2}\} \leqslant E_\omega\{\| G(t)(W(0) - R_N(0) - J_N(W(0) - R_N(0))\|_{0}^{2}\},$$

and

$$E_\omega\{\| V_N - J_N V_N \|^2\} \leqslant E_\omega\{\| W(0) - R_N(0) - J_N(W(0) - R_N(0))\|_{0}^{2}\}/2\lambda_0 .$$

Thus $E_\omega\{\| V_N - J_N V_N \|^2\}$ as well as $E_\omega\{\| J_N V_N \|^2\}$ is bounded for each $N$ uniformly in $T$. Combining these two results completes the proof of the lemma.

LEMMA 6. *$E_\omega\{\| R_N \|_{0}^{2}\}$ is bounded uniformly in $N$.*

*Proof.* By condition (H), Theorem 1, and Lemma 2,

$$E\{\| u_N \|^2\} \leqslant h^2 E\{| u_N |^2\} \leqslant h^2 b^2 E\{| W |^2\} \leqslant h^2 b^2 (C_1 T + C) = h^2 b^2 C_1 T + h^2 b^2 C_2,$$

where $C_1$ and $C_2$ are constants depending on the variance and mean of $W$. Since $u_N = R_N + V_N$,

$$E_\omega\{\| R_N \|^2\} \leqslant 2E_\omega\{\| u_N \|^2\} + 2E_\omega\{\| V_N \|^2\}.$$

Hence using the stationarity of $R_N$ and Lemma 5:

$$\begin{aligned} E_\omega\{\| R_N \|_{0}^{2}\} &= E_\omega\{\| R_N \|^2\}/T \\ &\leqslant 2(E_\omega\{\| u_N \|^2\} + E_\omega\{\| V_N \|^2\})/T \\ &\leqslant (2h^2 b^2 C_1 T + 2h^2 b^2 C_2 + 2a_N)/T \end{aligned}$$

Since $T$ can be arbitrarily large, $E_\omega\{\| R_N \|_{0}^{2}\} \leqslant 2h^2 b^2 C_1$, which completes the proof of the lemma.

LEMMA 7. *$\lim_{t\to\infty} E_\omega\{\| V_N(t)\|_{0}^{2}\} = 0$ uniformly in $N$.*

*Proof.* The proof is identical to that of Lemma 5 up to the points

$$E_\omega\{\| J_N V_N(t)\|_{0}^{2}\} \leqslant \exp(-2\lambda_0 t)\, E_\omega\{\| u(0) - R_N(0)\|_{0}^{2}\}$$

and

$$\begin{aligned} &E_\omega\{\| V_N(t) - J_N V_N(t)\|_{0}\} \\ &\quad\leqslant E_\omega\{\| G(t)(W(0) - R_N(0) - J_N(W(0) - R_N(0)))\|_{0}^{2}\}. \end{aligned}$$

But now from Lemma 6, $E_\omega\{\| R_N \|_{0}^{2}\}$ is uniformly bounded in $N$. This yields after some elementary manipulations,

$$E\{\| V_N(t)\|_{0}^{2}\} \leqslant C_3 \exp(-2\lambda_0 t),$$

where $C_3$ is a constant, which completes the proof of the lemma.

LEMMA 8. *$\lim_{N\to\infty} R_N$ exists in $H$.*

*Proof.* The proof is by contradiction. Assume $\lim_{N\to\infty} R_N$ does not exist in $H$. Then the stationarity of $R_N$ implies that for all $t$ there exists an $\epsilon$ such that

$$\inf_N \sup_{k,m>N} E_\omega\{\| R_k(t) - R_m(t)\|_0^2\} > 6\epsilon.$$

By Lemma 7 there exists an $s$ such that $E\{\| V_N \|_0^2\} < \epsilon$ for $t > s$.

$$\begin{aligned}
&\inf_N \sup_{k,m>N} \int_s^T E_\omega\{\| u_k(t) - u_m(t)\|_0^2\}\, dt \\
&\quad = \inf_N \sup_{k,m>N} \int_s^T E_\omega\{\| R_k + V_k - R_m - V_m \|_0^2\}\, dt \\
&\quad \geqslant \tfrac{1}{2} \inf_N \sup_{k,m>N} \int_s^T E_\omega\{\| R_k - R_m \|_0^2\}\, dt \\
&\quad\quad - \inf_N \sup_{k,m>N} \int_s^T E_\omega\{\| V_k - V_m \|_0^2\}\, dt \\
&\quad \geqslant 3\epsilon(T-s) - 2\epsilon(T-s) = \epsilon(T-s).
\end{aligned}$$

This implies that the mean square $\lim_{N\to\infty} u_N$ on $H$ does not exist.

However, Theorem 1 shows that $\lim_{N\to\infty} u_N = u$ on $B$ almost surely. Condition (H) ensures almost sure convergence on $H$. Finally, Lemmas 6 and 7 allow the bounded convergence theorem to be applied, yielding mean square convergence.

Thus there is a contradiction which means $\lim_{N\to\infty} R_N$ must exist on $H$.

Define $R = \lim_{N\to\infty} R_N$. Then $R$ is stationary since it is the limit of stationary processes.

Define $V(t) = u(t) - R(t)$.

LEMMA 9. $\operatorname{Lim}_{t\to\infty} E\{\| V(t)\|_0^2\} = 0$.

*Proof.* $V = u - R = \lim_{N\to\infty} u_n - R_N = \lim_{N\to\infty} V_N$ on $H$. By Lemma 7, $\lim_{t\to\infty} E\{\| V(t)\|_0^2\} = 0$ uniformly in $N$. This implies that $\lim_{t\to\infty} E_\omega\{\| V(t)\|_0^2\} = \lim_{t\to\infty} \lim_{N\to\infty} E\{\| V_N(t)\|_0^2\} = \lim_{N\to\infty} \lim_{t\to\infty} E\{\| V_N(t)\|_0^2\} = 0$.

The next step is to calculate the stationary distribution of $R_N(t, \omega)$ and $R(t, \omega)$.

*Notation* 1. The distribution of the random variable $u(\omega)$ on a space $S$ with respect to a measure $M$ has Radon–Nikodym derivative $\exp(-2f)$ means that for any $M$-measurable set $D \subset S$; $\operatorname{Prob}\{u(\omega) \in D\} = \int_D \exp(-2f)\, dM / \int_S \exp(-2f)\, dM$.

*Notation* 2. Fix $M$ as the Gaussian measure on $H_0$ with mean 0 and covariance $\int_0^\infty G^*(t)\, G(t)\, dt = -L^{-1}/2$. Using (L3) it is easy to show that $M(B_0) = 1$. Let $M_N$ be the Gaussian measure on $J_N B_0$ with mean 0 and covariance $\int_0^\infty J^*{}_N G^*(t) J_N G(t)\, dt = -J_N L^{-1}/2$.

LEMMA 10. *If $F$ is the Fréchet derivative of a functional $f$, then the stationary distribution of $u_N(t)$ and $R_N(t)$ has Radon–Nikodym derivative $\exp(-2f(J_N \cdot))$ with respect to $M$.*

*Proof.* The method of the proof is similar to that used in [1, Appendix]. A sketch of the proof is the following. $J_N u_N$ satisfies a finite-dimensional Itô equation. On a ball in the finite-dimensional space $J_N B_0$ the probability density of $J_N u_N$ satisfies a parabolic partial differential equation. The equation has a unique solution because of the monotonicity and continuity of its coefficients combined with the maximum principle. With suitable boundary conditions the distribution whose Radon–Nikodym derivative is $\exp(-2f(J_N \cdot))$ with respect to $M_N$ restricted to the ball is the stationary distribution for the equation.

The next step is to show that as the ball expands the probability of any path of $J_n u_n$ hitting the boundary goes to 0. This follows from the uniform monotonicity of $F$ and condition (L1). Hence the boundary conditions become negligible as the radius of the ball goes to $\infty$. Thus the stationary distribution of $J_N u_N$ on $J_N R_0$ has Radon–Nikodym derivative $\exp(-2f(J_N \cdot))$ with respect to $M_N$. Note that Lemma 4 guarantees uniqueness of the stationary distribution.

Finally $u_N = J_N u_N + W - J_N W$ and since $J_N u_N$ and $W - J_N W$ are independent random variables it is easy to verify that the stationary distribution of $u_N$ on $B_0$ has Radon–Nikodym derivative $\exp(-2f(J_N \cdot))$ with respect to $M$. This completes the proof of Lemma 10.

LEMMA 11. *The stationary distribution of $R(t)$ has Radon–Nikodym derivative $\exp(-2f(\cdot))$ with respect to $M$ on $H_0$.*

*Proof.* Since $\lim_{N\to\infty} R_N = R$ on $H_0$, one must show that $\lim_{N\to\infty} \int_{H_0} |\exp(-2f(J_N \cdot)) - \exp(-2f(\cdot))|\, dM = 0$. Note that $\lim_{N\to\infty} f(J_N \cdot) = f(\cdot)$ on a set with $M$ measure equal to 1. Let $f(0) = 0$ with no loss of generatlity. Then the monotonicity of $F$ ensures that $f$ is nonnegative and $\exp(-2f) \leqslant 1$.

The Lebesque bounded convergence theorem can now be applied to show that the limit of the integral is equal to 0, completing the proof of Lemma 11.

THEOREM 3. *Given the equation*

$$u(t) = -\int_0^t G(t-s)\, F(u(s))\, ds + W(t)$$

*with all the conditions of Theorem 1 and $F$ the Fréchet derivative of a functional $f$. Then the unique solution $u(t, \omega)$ has the form $u(t, \omega) = R(t, \omega) + V(t, \omega)$, where $\lim_{t\to\infty} E_\omega\{\| V(t, \omega)\|_0^2\} = 0$ almost surely and $R(t, \omega)$ is a stationary process with a distribution on $B_0$ having Radon–Nikodym derivative $\exp(-2f)$ with respect to $M$, which is a Gaussian measure of mean 0 and covariance operator $\int_0^t G^*(t)\, G(t)\, dt = -L^{-1}/2$.*

*Proof.* This follows immediately from Lemmas 4, 8, 9 and 11.

## APPENDIX

One concrete example for which Theorem 3 applies is

$$u_t = u_{xx} - P(u(\chi)) + \alpha(\chi, t, \omega)$$

with $u(0, t) = u(1, t) = 0$ and $u(\chi, 0) \in L^p(0, 1)$.

$\alpha(\chi, t, \omega)$ is a *white noise* in $\chi$ and $t$, i.e., a Gaussian process with mean 0 such that

$$E\{\alpha(\chi, t, \omega)\, \alpha(y, s, \omega)\} = \delta(\chi - y)\, \delta(t - s).$$

$P$ is a nonconstant polynomial in $u$ of degree $p - 1$ and has a nonnegative derivative. It is easy to verify that the conditions of Theorem 3 are satisfied and thus all of the conclusions follow. The stationary distribution corresponds to that of a Markov process in the parameter $\chi$. (This follows heuristically from the local nature of the functional $P$).

The idea motivating this paper is that it may be possible to obtain information about the limiting process in $\chi$ by studying the evolving process $u(\chi, t)$. For example, in [1, Theorem 4] a perturbation expansion is derived for $u(\chi, t)$ and is shown to be asymptotic uniformly in $t$, thus giving an expanison for the limiting process.

It is hoped that the results of the present paper can be generalized to the following cases.

(A) $P$ is no longer monotone but instead is a polynomial of odd degree with a derivative bounded below. This extension does not appear to be too difficult.

(B) The space variable is two dimensional. In this case $P(u)$ must be replaced by a Wick polynomial $:P(u):$. The limiting process would then be a Markov field on a two-dimensional domain. See [4].

(C) The space variable is three dimensional. In this case even a Wick polynomial at a fixed time would not be defined but an integral representation such as (3) might be meaningful. See [5].

Any information obtained about the limiting distributions of (B) and (C) would be of great interest because of the relationship between Markov fields and quantum fields. See [6].

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